# Bayesian Analysis of the Coin Bias

It is possible to determine the probability density for the coin's bias $H$ using a Bayesian approach. In this example, we assume no prior knowledge, so we start with a uniform prior for $H$ (i.e., every bias between 0 and 1 is equally likely). We flipped the coin 10 times and observed 7 heads.

## 1. Setting Up the Problem

- **Unknown Parameter:** $H$ is the probability that the coin lands heads.
- **Data:** The coin was flipped 10 times and 7 heads were observed.
- **Prior Assumption:** With no prior knowledge, we assume a uniform prior for $H$ over the interval $[0, 1]$:

$$p(H) = 1 \quad \text{for } 0 \leq H \leq 1.$$

## 2. The Likelihood Function

The likelihood of obtaining 7 heads in 10 flips given a bias $H$ is modeled by the binomial distribution:

$$p(\text{data} \mid H) = \binom{10}{7} H^7 (1 - H)^3,$$

where $\binom{10}{7}$ is the binomial coefficient. Since this coefficient is independent of $H$, it will be absorbed into the normalization constant when we form the posterior.

## 3. Applying Bayes' Theorem

Bayes' theorem tells us:

$$p(H \mid \text{data}) \propto p(\text{data} \mid H)\, p(H).$$

Given our uniform prior, this becomes:

$$p(H \mid \text{data}) \propto H^7 (1 - H)^3.$$

This functional form corresponds to a **Beta distribution**. In general, the Beta distribution is given by:

$$\text{Beta}(H; \alpha, \beta) \propto H^{\alpha - 1} (1 - H)^{\beta - 1}.$$

Comparing exponents, we have:

$$\alpha - 1 = 7 \quad \Longrightarrow \quad \alpha = 8, \quad \text{and} \quad \beta - 1 = 3 \quad \Longrightarrow \quad \beta = 4.$$

Thus, the posterior distribution is:

$$p(H \mid \text{data}) = \text{Beta}(H; 8, 4).$$

## 4. Normalization Using the Beta Function

The normalized Beta distribution is given by:

$$p(H \mid \text{data}) = \frac{H^7 (1 - H)^3}{B(8, 4)},$$

where the **Beta function** $B(\alpha, \beta)$ is defined as:

$$B(\alpha, \beta) = \int_0^1 t^{\alpha - 1} (1 - t)^{\beta - 1}\, dt = \frac{\Gamma(\alpha)\, \Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

The **Gamma function** $\Gamma(n)$ generalizes the factorial function with $\Gamma(n) = (n-1)!$ for positive integers $n$.

For our parameters:

$$B(8,4) = \frac{\Gamma(8)\,\Gamma(4)}{\Gamma(12)} = \frac{7! \cdot 3!}{11!}.$$

Evaluating:

$$7! = 5040, \quad 3! = 6, \quad 11! = 39916800,$$

we get:

$$B(8,4) = \frac{5040 \times 6}{39916800} = \frac{30240}{39916800} = \frac{1}{1320}.$$

Thus, the normalized probability density is:

$$p(H \mid \text{data}) = 1320\, H^7 (1-H)^3 \quad \text{for } 0 \le H \le 1.$$

## 5. Calculating the Probability Over an Interval

To find the probability that $H$ lies in an interval $[a,b]$ (with $0 \le a < b \le 1$), integrate the density over that interval:

$$P(a \le H \le b) = \int_a^b 1320\, H^7 (1-H)^3 \, dH.$$

This integral is often expressed in terms of the **regularized incomplete Beta function** $I_x(\alpha,\beta)$, defined as:

$$I_x(\alpha,\beta) = \frac{1}{B(\alpha,\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\, dt.$$

Thus, for our case:

$$P(a \le H \le b) = I_b(8,4) - I_a(8,4).$$

## 6. Evaluating the Density at Specific Values

The posterior density is:

$$p(H \mid \text{data}) = 1320\, H^7 (1-H)^3.$$

We now calculate the value of this density at the following specific points:

### At $H = 0.1$

$$p(0.1) = 1320 \times (0.1)^7 \times (0.9)^3.$$

Calculations:

$$(0.1)^7 = 10^{-7} = 1 \times 10^{-7},$$
$$(0.9)^3 \approx 0.729,$$

so,

$$p(0.1) \approx 1320 \times 1 \times 10^{-7} \times 0.729 \approx 9.62 \times 10^{-5}.$$

### At $H = 0.3$

$$p(0.3) = 1320 \times (0.3)^7 \times (0.7)^3.$$

Calculations:

$$(0.3)^7 \approx 0.0002187,$$
$$(0.7)^3 \approx 0.343,$$

so,

$$p(0.3) \approx 1320 \times 0.0002187 \times 0.343 \approx 0.099.$$

### At $H = 0.5$

$$p(0.5) = 1320 \times (0.5)^7 \times (0.5)^3.$$

Calculations:

$$(0.5)^7 = \frac{1}{128} \approx 0.0078125,$$
$$(0.5)^3 = \frac{1}{8} = 0.125,$$

so,

$$p(0.5) \approx 1320 \times 0.0078125 \times 0.125 = 1320 \times \frac{1}{1024} \approx 1.289.$$

### At $H = 0.7$

$$p(0.7) = 1320 \times (0.7)^7 \times (0.3)^3.$$

Calculations:

$$(0.7)^7 \approx 0.0823543,$$
$$(0.3)^3 = 0.027,$$

so,

$$p(0.7) \approx 1320 \times 0.0823543 \times 0.027 \approx 2.936.$$

(Note: This value is near the mode of the distribution, which occurs at $H = \frac{\alpha-1}{\alpha+\beta-2} = \frac{7}{10} = 0.7$.)

### At $H = 0.9$

$$p(0.9) = 1320 \times (0.9)^7 \times (0.1)^3.$$

Calculations:

$$(0.9)^7 \approx 0.4783,$$
$$(0.1)^3 = 0.001,$$

so,

$$p(0.9) \approx 1320 \times 0.4783 \times 0.001 \approx 0.631.$$

## Final Summary

- **Posterior Density:**
$$p(H \mid \text{data}) = 1320\, H^7 (1-H)^3, \quad \text{for } 0 \leq H \leq 1.$$
- **Probability over an Interval $[a, b]$:**
$$P(a \leq H \leq b) = \int_a^b 1320\, H^7 (1-H)^3 \, dH = I_b(8,4) - I_a(8,4),$$
where $I_x(8,4)$ is the regularized incomplete Beta function.
- **Density Values at Specific $H$:**
$$p(0.1) \approx 9.62 \times 10^{-5},$$
$$p(0.3) \approx 0.099,$$
$$p(0.5) \approx 1.289,$$
$$p(0.7) \approx 2.936,$$
$$p(0.9) \approx 0.631.$$

This derivation explains how to obtain the probability density function for the coin bias $H$ after observing 7 heads in 10 flips, starting from a uniform prior and using Bayesian analysis.